Chatter Stability Characterization of a Plastic End-Milling CNC Machine

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Abstract

The desire to carry out this work arose from an observation during a practical work on a typical plastic end milling CNC machine. It was noticed that under certain conditions of cutting, operation of the machine became noisy with increasing depth of cut and eventual perforation of workpiece therefore the basic aim is to generate stability characterization of the machine in the form of a chart on the plane of cutting parameters on which stable operation is demarcated from the unstable operation . In modelling this machine, a slot creating mode of operation is used since the machine is mainly used for creating logos which are basically collection of slots. The significance of the resulting stability chart lies in the result that the cause of the aforementioned noisy operation is due to unstable parameter combination. For example a laboratory operation at spindle speed of 1500rpm and depth of cut of 1.5mm was noisy while that at spindle speed of 1500rpm and depth of cut of 1mm was serene. The stability chart generated for the system thus shows close agreement with both practice and theory. A unique impact this work will have on the reading community will be in the area of validity of the resulting stability chart on the basis of MATLAB dde23 numerical simulation. The parameters of the end milling process are; tool mass $m = 0.0431\text{kg}$, tool natural frequency $\omega_n = 5700\ \text{rad/sec}$, damping factor $\xi = 0.02$ and workpiece cutting coefficient$C = 3.5 \times 10^7\text{Nm}^{\frac{-7}{4}}$.

Keywords: chatter, time history, Fargue approximation, Floquet theory, bifurcation

1. Introduction

Components of high dimensional integrity are in ever increasing need. Machine tools such as Lathe and Milling machines are needed for production of such components. They would not perform effectively under highly disturbed situations thus the need for vibration control in such machines. Achieving good surface finish and high productivity are two opposed demands in machining operation. This means that ascertaining safe operation range for good product, improved tool life and design of machine tools is necessary.

A typical machining process of major importance is the end-milling in which a machined surface that is at right angle with the cutter axis results as shown Figure1. End milling cutters equipped with shanks for mounting on the spindle are utilized for end milling.

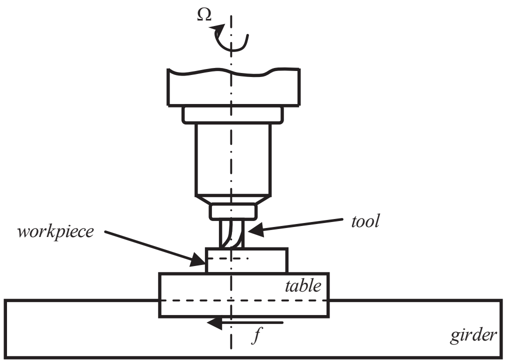


Figure1. End-milling

Machine tool vibration is basically called chatter. Chatter invariably results whenever there is dynamic interaction between the tool and the workpiece (project) of a milling process. Forced, self-exited and damped natural vibrations combine to compound the dynamics of milling process. The forced vibration component is a periodic disturbance that stems from regular engagement and dis-engagement of tool and workpiece. Regenerative effect is underscored as the major cause of the self-exited vibrations (mechanical chatter) in machining (Stepan et al, 2003; Insperger, 2002). Regenerative effect is a concept used to explain the sustained vibration occurring during machining as resulting from cutting force variation due to vibration induced surface waviness. Arnold first suggested regenerative effects as the potential cause of chatter and is now arguably considered the cause of detrimental type of machine tool vibration (Davies et al, 1999). The effect of delayed position on the present position of the tool causes modelling of regenerative vibrations to result in delay differential equations (DDEs). Major milestones have been made in the area of milling regenerative vibrations. Some of the most popular achievements in contemporary milling machine vibration studies are stability charts. Among the Various methods utilized in their works in tracking the milling stability boundary are; the finite element in time (Insperger et al, 2003; Butcher and Mann, 2011), Chebyshev Polynomials (Butcher and Mann, 2011), semi-discretization (Insperger and Stepan, 2002) and fargue-type approximation (Insperger, 2002; Insperger and Stepan, 2000). The aim in this work is to adapt some of these achievements in the stability characterization of a Perspex or wood end milling CNC machine estimated in (Ozoegwu, 2011) to have the following modal parameters; mass $m = 0.0431kg$, Natural frequency $\omega_n = 5700\ rad/sec$, and damping factor $\xi = 0.02$ for systematic operation. As part of contribution of this work the resulting stability chart is validated with numerical integration of the governing DDE at selected points of the parameter space by MATLAB dde23.

2. Mathematical Model

In the dynamical model shown in figure2, the tool is given a spindle speed Ω in revolutions per minute while the workpiece has a prescribed feed velocity v imparted on it via the worktable. The model being considered is a milling tool with three teeth creating a slot through a workpiece. The parameters of the milling process as depicted on the dynamical model are; m mass of tool, c the equivalent viscous damping coefficient modelling the hysteretic damping of the tool system and k the stiffness of the tool system. These modal parameters could be extracted from plot of the tool frequency response function in a scheme of experimental modal analysis. Figure2 is a single degree of freedom vibration model of an end milling tool. Most encountered resonance in machining involves the fundamental natural frequency thus single degree of freedom vibration is satisfactory when it is well separated from the higher frequencies (Stepan, 1998). The wavy regenerative machined surface that sustains chatter vibration is shown enlarged on figure 2.

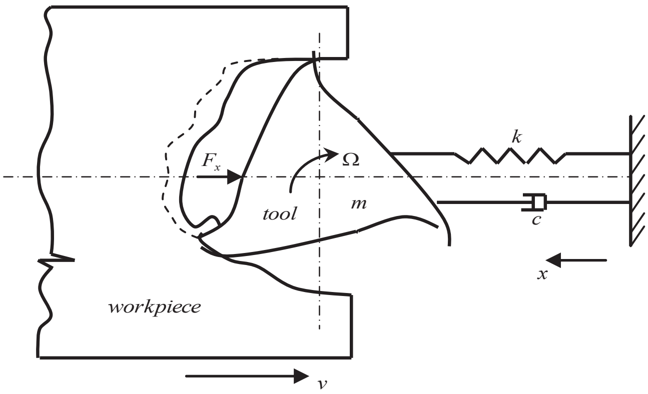


Figure2. Dynamical model of milling

The free-body diagram for the tool dynamics is as shown in figure3.

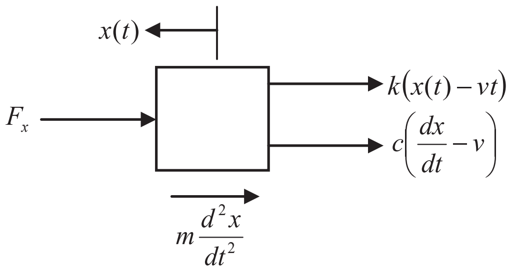


Figure3. Free-body diagram of tool dynamics

The differential equation governing the motion of the tool as seen from the free-body diagram is

$$m\ddot{x}(t) + c[\dot{x}(t) - vt] + k[x(t) - vt] + F_x = 0 \qquad (1)$$

A tool-workpiece disposition as shown in figure4 is considered for the jth tooth of the tool. The x −component of cutting force for the tool thus becomes

$$F_x(t) = \sum_{j=1}^{N} g_j(t)\left[F_{norm,j}(t)\sin\theta_j(t) + F_{tang,j}(t)\cos\theta_j(t)\right] \qquad (2)$$

N is the number teeth on the milling tool indexed with the values j =1, 2, 3......N. The instantaneous angular position of a tooth j is $\theta_j(t)$. In this work $\theta_j(t)$ is measured clockwise relative to the negative y −axis to give

$$\theta_j(t) = \left(\frac{\pi\Omega}{30}\right)t + (j-1)\frac{2\pi}{N} + \alpha \qquad (3)$$

Where α is the initial angular position of the tooth indexed 1. Screen or switching function for the jth tooth $g_j(t)$ could either have the values 1 or 0 depending on whether the tooth is active or not. Since the tool is creating a slot on the workpiece as shown in figure2, the start and end angles will have the values $\theta_s = 0^\circ$ and $\theta_e = 180^\circ$ respectively. Under this operating condition it becomes clear from the workpiece-tool disposition of figure4 that the screen function could be expressed thus;

$$g_j(t) = \frac{1}{2}\left\{1 + sgn\left[sin\left(\theta_j(t)\right)\right]\right\} \tag{4}$$

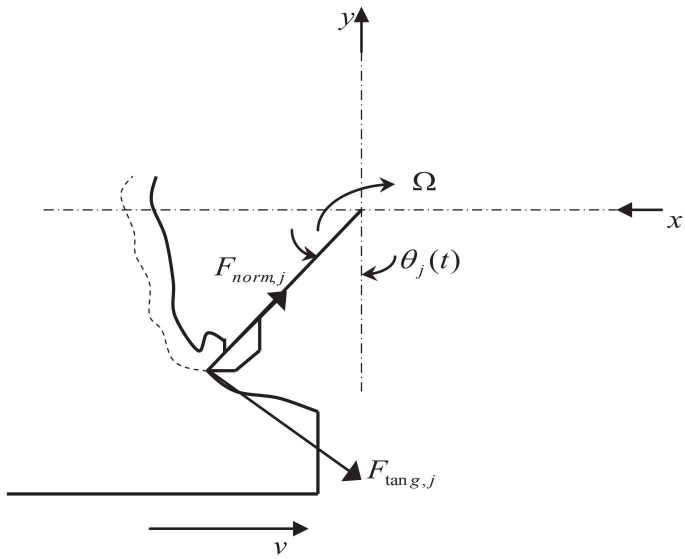


Figure4. Milling tooth workpiece disposition

The tangential cutting force for the j tooth is given by the non-linear law (Insperger, 2002)

$$F_{tang,j}(t) = Cw\{f_a sin\theta_j(t)\}^{\gamma} \tag{5}$$

where w is depth of cut, C is the cutting coefficient associated with the workpiece which is assumed to have the value $C = 3.5 \times 10^7\,\mathrm{Nm}^{\frac{-7}{4}}$ for Perspex for reasons given in (Ozoegwu, 2011), f_a is the actual feed and γ is an exponent that is usually less than one having a value of $^3/_4$ for the three-quarter rule. It is written in (Insperger, 2002) that empirical relationship connects the milling tangential and normal cutting forces in the works of Balint, Bali and Tlusty according to the equation

$$F_{norm,j}(t) = 0.3F_{tang,j}(t) \tag{6}$$

The actual feed rate f_a is the difference between present and one period delayed position of tool, thus

$$f_a = x(t) - x(t-\tau) \tag{7}$$

Equations (5), (6) and (7) taken together give

$$F_x(t) = wq(t)[x(t) - x(t-\tau)]^{\gamma} \tag{8}$$

where $q(t) = \sum_{j=1}^{n} g_j(t)\, C sin^{\gamma}\theta_j(t)\left[0.3 sin\theta_j(t) + cos\theta_j(t)\right]$ is a $\tau(=\frac{60}{N\Omega})$ periodic function. Introducing Equation (8) into the equation of motion of the tool system (1) gives

$$m\ddot{x}(t) + c[\dot{x}(t) - vt] + k[x(t) - vt] + wq(t)[x(t) - x(t-\tau)]^{\gamma} = 0 \quad (9)$$

Suppose the motion of the tool is assumed to be a linear superposition of prescribed feed motion vt, tool response with period τ due to periodic force of tool-workpiece interaction $x_t(t)$ and perturbation $z(t)$ (Insperger, 2002) mainly due to regenerative effects then

$$x(t) = vt + x_t(t) + z(t) \quad (10)$$

Substitution of equation (10) into equation (9) givess

$$m\ddot{x}_t(t) + c\dot{x}_t(t) + kx_t(t) + m\ddot{z}(t) + cz(t) + kz(t) = -wq(t)\left[v\tau + \left(z(t) - z(t-\tau)\right)\right]^{\gamma} \quad (11)$$

Without perturbation (that is $z = z(t-\tau) = 0$), equation (11) simplifies to

$$m\ddot{x}_t(t) + c\dot{x}_t(t) + kx_t(t) = -wq(t)[v\tau]^{\gamma} \quad (12)$$

Equation (12) means that equation (11) becomes

$$m\ddot{z}(t) + cz(t) + kz(t) = wq(t)[v\tau]^{\gamma} - wq(t)\left[v\tau + \left(z(t) - z(t-\tau)\right)\right]^{\gamma} \quad (13)$$

Put in Taylor series about $v\tau$ and linearizing equation (13) becomes

$$m\ddot{z}(t) + cz(t) + kz(t) = -wh(t)\left(z - z(t-\tau)\right) \quad (14)$$

Where $h(t) = \gamma(v\tau)^{\gamma-1}q(t)$ is the time-varying specific force variation (Insperger, 2002).

Equation (14) is re-written with the following compact notations; $z(t) = z$ and $z(t-\tau) = z_\tau$ to give a form similar to damped delayed Mathieu equation (15) which is the equation of regenerative vibration of the system.

$$\ddot{z} + 2\xi\omega_n\dot{z} + \left(\omega_n^2 + \frac{wk(t)}{m}\right)z = \frac{wk(t)}{m}z_\tau \quad (15)$$

With the substitutions $y_1 = z \; and \; y_2 = \dot{z}$ made, equation (15) could be put in state differential equation form as

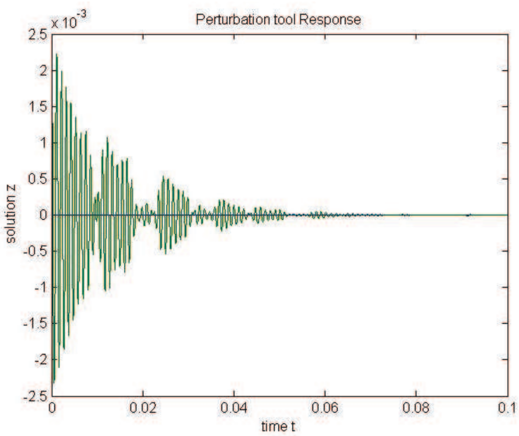
$$\begin{Bmatrix}\dot{y}_1\\ \dot{y}_2\end{Bmatrix} = \begin{bmatrix}0 & 1\\ -\left(\omega_n^2 + \frac{wk(t)}{m}\right) & -2\xi\omega_n\end{bmatrix}\begin{Bmatrix}y_1\\ y_2\end{Bmatrix} + \begin{bmatrix}0 & 0\\ \frac{wk(t)}{m} & 0\end{bmatrix}\begin{Bmatrix}y_{1,\tau}\\ y_{2,\tau}\end{Bmatrix} \quad (16)$$

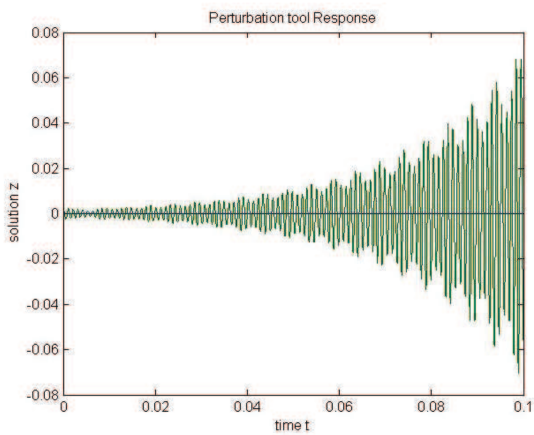
Where $y_{i,\tau} = y_i(t-\tau)$ for $= 1 \; and \; 2$.

The natural frequency and damping ratio of the tool system are given in terms of modal parameters $k, m \; and \; c$ respectively as $\omega_n = \sqrt{\frac{k}{m}} \; and \; \xi = \frac{c}{2\sqrt{mk}}$. These modal parameters are easily extracted from experimental plot of the tool frequency response function $R(\omega) = \frac{X}{F} = \frac{1}{\sqrt{(k-\omega^2 m)^2 + \omega^2 c^2}}$ for forced single degree of freedom vibration. As shown in figure2 the number of teeth N on the tool considered is three. The tool tooth indexed 1 is assumed to have an initial angular position $\alpha = 0$ at the beginning of milling feed, then equation (3) gives: $\theta_1(t) = \left(\frac{\pi\Omega}{30}\right)t$, $\theta_2(t) = \left(\frac{\pi\Omega}{30}\right)t + \frac{2\pi}{3}$ and $\theta_3(t) = \left(\frac{\pi\Omega}{30}\right)t + \frac{4\pi}{3}$ giving rise to specific force variation becoming

$$h(t) = C\gamma(v\tau)^{\gamma-1}\Big\{\tfrac{1}{2}\Big\{1+sgn\Big[\sin\Big(\Big(\tfrac{\pi\Omega}{30}\Big)t\Big)\Big]\Big\}sin^{\gamma}\Big(\Big(\tfrac{\pi\Omega}{30}\Big)t\Big)\Big[0.3sin\Big(\Big(\tfrac{\pi\Omega}{30}\Big)t\Big)+cos\Big(\Big(\tfrac{\pi\Omega}{30}\Big)t\Big)\Big]+\tfrac{1}{2}\Big\{1+ sgn\Big[\sin\Big(\Big(\tfrac{\pi\Omega}{30}\Big)t+\tfrac{2\pi}{3}\Big)\Big]\Big\}sin^{\gamma}\Big(\Big(\tfrac{\pi\Omega}{30}\Big)t+\tfrac{2\pi}{3}\Big)\Big[0.3sin\Big(\Big(\tfrac{\pi\Omega}{30}\Big)t+\tfrac{2\pi}{3}\Big)+cos\Big(\Big(\tfrac{\pi\Omega}{30}\Big)t+\tfrac{2\pi}{3}\Big)\Big]+ \tfrac{1}{2}\Big\{1+sgn\Big[\sin\Big(\Big(\tfrac{\pi\Omega}{30}\Big)t+\tfrac{4\pi}{3}\Big)\Big]\Big\}sin^{\gamma}\Big(\Big(\tfrac{\pi\Omega}{30}\Big)t+\tfrac{4\pi}{3}\Big)\Big[0.3sin\Big(\Big(\tfrac{\pi\Omega}{30}\Big)t+\tfrac{4\pi}{3}\Big)+cos\Big(\Big(\tfrac{\pi\Omega}{30}\Big)t+\tfrac{4\pi}{3}\Big)\Big]\Big\} \tag{17}$$

The damped delayed Mathieu equation (16) can be solved upon substitution of equation(17). Making use of the parameters of the system; $m = 0.0431kg$, $\omega_n = 5700\ rad/sec$, $\xi = 0.02$, $C = 3.5\times 10^7\,\mathrm{Nm}^{\frac{-7}{4}}$ and $v = 0.0025m/s$, two MATLAB dde23 sample time histories of the system based on equation (16) together with the determining cutting parameters (Ω and w) are as shown in figure5. It is seen that the response of the tool at an operating condition of spindle speed $\Omega = 3000\mathrm{rpm}$ and depth of cut $w = 1\mathrm{mm}$ is asymptotically stable while that at a spindle speed $\Omega = 4000\mathrm{rpm}$ for same depth of cut is unstable. The perturbation history used is $\{y_1(t),\ \ y_2(t)\}^T = \{0,\ \ 0.0025\}^T$ where $t\epsilon[-\tau, 0]$. It is found in (Ozoegwu, 2011) through numerical simulation of MATLAB dde23 that condition of stability and instability is determined entirely by cutting parameter combination of spindle speed and depth of cut. This means that any arbitrary choice of perturbation history whether determinist or stochastic will not influence stability result of MATLAB dde23.





$\Omega = 3000\text{rpm}, w = 1\text{mm}$

$\Omega = 4000\text{rpm}, w = 1\text{mm}$

Figure 5. Milling tool perturbation time histories

3. Stability Analysis of the End-Milling Process

Systematic selection process for cutting parameters that will result in good surface texture and integrity can only result from proper mathematical modelling and stability analysis of milling process. Stable milling process is needed for surface accuracy and integrity in which case the possibility of failure by fatigue, corrosion and wear of a machined element is reduced by avoiding adverse alteration of machined surface. Stable milling operation could then be deemed to be a form of proactive milling machine tool maintenance since tool-breaking and machine damaging vibrations are jettisoned.

Stability investigation of the system presently considered is based on equation (15). Via Fargue-type approximation, periodic delay-differential equation (DDE) could be transformed into periodic ordinary differential equation(ODE). The resulting ODE is investigated based on the Floquet theory for stability analysis of periodic ODE. As proved by Fargue, the following holds (Insperger, 2002).

$$\lim_{n\to\infty}\int_{-\infty}^{0} w(\vartheta)z(t+\vartheta)d\vartheta = z(t-\tau) = z_\tau \tag{18}$$

Where $w(\vartheta) = (-1)^n \frac{n^{n+1}}{\tau^{n+1}n!}\vartheta^n e^{n\vartheta/\tau}$ is the Fargue weight function. In light of equation (18), equation (15) becomes

$$\ddot{z} + 2\xi\omega_n\dot{z} + \left(\omega_n^2 + \frac{wk(t)}{m}\right)z = \frac{wk(t)}{m}\left[\lim_{n\to\infty}\int_{-\infty}^{0} w(\vartheta)z(t+\vartheta)d\vartheta\right] \tag{19}$$

Fargue approximation becomes $z_\tau = z(t-\tau) \approx \int_{-\infty}^{0} w(\vartheta)z(t+\vartheta)d\vartheta$ For appreciably high finite n, thus good results still results from the equation

$$\ddot{z} + 2\xi\omega_n\dot{z} + \left(\omega_n^2 + \frac{wk(t)}{m}\right)z = \frac{wk(t)}{m}\int_{-\infty}^{0} w(\vartheta)z(t+\vartheta)d\vartheta \tag{20}$$

It is then seen that n is the Fargue approximation parameter. State variables of form

$$y_r = \int_{-\infty}^{0}\left[\frac{n!}{[n-(r-3)]!}\frac{w(\vartheta)}{\vartheta^{r-3}}\right]z(t+\vartheta)d\vartheta \tag{21}$$

where$r = 3, 4, 5, \ldots\ldots .(n+3)$, enables the transformation of equation (20) into the state vector equation

$$
\begin{aligned}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -\left(\omega_n^2 + \frac{wk(t)}{m}\right) y_1 - 2\xi\omega_n y_2 + \frac{wk(t)}{m} y_3 \\
\dot{y}_3 &= -\frac{n}{\tau} y_3 - y_4 \\
\dot{y}_4 &= -\frac{n}{\tau} y_4 - y_5 \\
\dot{y}_5 &= -\frac{n}{\tau} y_5 - y_6 \\
&\vdots \\
&\vdots \\
\dot{y}_{n+3} &= (-1)^n \frac{n^{n+1}}{\tau^{n+1}} y_1 - \frac{n}{\tau} y_{n+3}
\end{aligned} \tag{22}
$$

Where the $(n+3) \times (n+3)$ periodic coefficient matrix $\boldsymbol{A}(t)$ is given thus

$$
\boldsymbol{A}(t) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & \dots & 0 \\
-\left(\omega_n^2 + \frac{wk(t)}{m}\right) & -2\xi\omega_n & \frac{wk(t)}{m} & 0 & 0 & \dots & 0 \\
0 & 0 & -\frac{n}{\tau} & -1 & 0 & \dots & 0 \\
0 & 0 & 0 & -\frac{n}{\tau} & -1 & \dots & 0 \\
0 & 0 & 0 & 0 & -\frac{n}{\tau} & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \dots & -\frac{n}{\tau} & -1 \\
(-1)^n \frac{n^{n+1}}{\tau^{n+1}} & 0 & 0 & 0 & \dots & 0 & -\frac{n}{\tau}
\end{bmatrix} \tag{23}
$$

According to the Floquet theory the time varying periodic ODE (43) has a solution of form;

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}(t)\,\boldsymbol{y}(0) \tag{24}$$

Where $\boldsymbol{\Phi}(t)$ is the *fundamental matrix* of the system which has been proven by Floquet to have the form

$$\boldsymbol{\Phi}(t) = \mathbf{P}(t) e^{\boldsymbol{B}t} \tag{25}$$

Where $\boldsymbol{B}$ is a constant matrix. $\mathbf{P}(t)$ has the following properties; periodicity such that if T is the principal period then $\mathbf{P}(t) = \mathbf{P}(t+T)$ and initial condition of identity matrix such that $\mathbf{P}(0) = \boldsymbol{I}$. These two properties imply that

$$\boldsymbol{\Phi}(T) = e^{\boldsymbol{B}T} \tag{26}$$

$e^{\boldsymbol{B}T}$ is called the *principal* or *monodromy* or *Floquet transition matrix*. At time of one period after the initial condition equation (24) becomes

$$\boldsymbol{y}(T) = \boldsymbol{\Phi}(T)\,\boldsymbol{y}(0) \tag{27}$$

The eigenvalues of $\boldsymbol{\Phi}(T)$ designated μ_i and eigenvalues of $\boldsymbol{B}$ designated λ_i are called the *characteristic multipliers* and *characteristic exponents* respectively. It can be seen that the relationship between characteristic *multipliers* μ_i and corresponding *characteristic exponents* λ_i is

$$\mu_i = e^{\lambda_i T} \tag{28}$$

If a characteristic exponent is given as $\lambda_i = \sigma + j\omega$ then $\mu_i = e^{\sigma T} e^{j\omega T}$. It follows that

$$|\mu_i| = e^{\sigma T} \tag{29}$$

Stability of equation (15) requires all characteristic exponents to have negative real parts, that is $\sigma < 0$. From equation (29) the stability criterion for the system can also be stated to mean that all characteristic multipliers have modulus less than one, that is $|\mu_i| < 1$. It can be implied from equation (29) that at critical operating conditions when there exist characteristic multipliers such that $|\mu_i| = 1$ that the corresponding characteristic exponents are pure imaginary. This is a loss of stability condition that occurs when a least one characteristic multiplier is moving out of a unit circle centred on the origin of the complex plane by either period two bifurcation or secondary Hopf bifurcation(Insperger, 2002).

The Floquet fundamental matrix is difficult to achieve thus numerical approximations of the monodromy matrix are used in stability analysis of periodic ODE's. A typical method of achieving an estimate of the principal matrix is by piecewise constant approximation of the of the time-varying coefficient matrix $\boldsymbol{A}(t)$ (Insperger, 2002). This involves dividing the principal period T of the system into k time intervals $[t_i, t_{i+1}]$ where $i = 0, 1, 2, \dots \dots \dots (k-1)$ and approximating the coefficient matrix of the system $\boldsymbol{A}(t)$ at the midpoint of each of the time intervals. In this work equal time intervals $\Delta t = t_{i+1} - t_i$ are used such that

$$\boldsymbol{A}\left(t_i + \frac{\Delta t}{2}\right) = \boldsymbol{A}_i = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ -\left(\omega_n^2 + \frac{wk\left(t_i+\frac{\Delta t}{2}\right)}{m}\right) & -2\xi\omega_n & \frac{wk\left(t_i+\frac{\Delta t}{2}\right)}{m} & 0 & 0 & \dots & 0 \\ 0 & 0 & -\frac{n}{\tau} & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & -\frac{n}{\tau} & -1 & \dots & 0 \\ 0 & 0 & 0 & 0 & -\frac{n}{\tau} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\frac{n}{\tau} & -1 \\ (-1)^n \frac{n^{n+1}}{\tau^{n+1}} & 0 & 0 & 0 & \dots & 0 & -\frac{n}{\tau} \end{bmatrix} \tag{30}$$

Also $\Delta t = \frac{T}{k}$ is the constant time interval. By coupling the piecewise constant approximations of the coefficient matrix$\boldsymbol{A}_i$, an approximate Floquet transition matrix becomes

$$\boldsymbol{\Phi}(T) = \prod_{i=k-1}^{0} e^{A_i \Delta t} \tag{31}$$

Stability of milling process described by equation (15) could be derived by investigating the nature of the eigenvalues of $\boldsymbol{\Phi}(T)$ for various parameter operations of the system. Any combination of cutting parameters that result in any of the eigenvalues of the approximate monodromy matrix becoming greater than one is a chatter operating point.

A critical look on the last row of the coefficient matrix of equation (30) shows that maximum row sum norm approaches infinity as both n and Ω increase. This suggests ill-conditioning if n is high enough to be useful in the Fargue approximation. This type of numerical problem could be avoided by making use of the dimensionless time $\tilde{t} = \frac{n}{\tau} t$ (Insperger, 2002) to obtain an approximate piecewise constant matrix for the system as

$$\boldsymbol{A}\left(t_i+\frac{\Delta t}{2}\right)=\boldsymbol{A}_i=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ -\left(\frac{\tau}{n}\right)^2\left(\omega_n^2+\frac{wk\left(t_i+\frac{\Delta t}{2}\right)}{m}\right) & -2\xi\omega_n\left(\frac{\tau}{n}\right) & \frac{wk\left(t_i+\frac{\Delta t}{2}\right)}{m}\left(\frac{\tau}{n}\right)^2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & -1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & -1 \\ (-1)^n & 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix} \tag{32}$$

Row sum norm of interest now stems from row 2 of (32). It is clear that as n and Ω increase that the ratio $\frac{\tau}{n}$ decreases thus if the stability investigation starts from a-not-too low spindle speed the computer computations involving the coefficient matrix will be well-conditioned while requiring a much lower memory space. The estimation of the Floquet transition matrix utilizing equal time intervals becomes

$$\boldsymbol{\Phi}\left(\frac{n}{\tau}\mathrm{T}\right)=\prod_{i=k-1}^{0} e^{A_i\frac{n}{\tau}\Delta t} \tag{33}$$

Equations (32) and (33) are the basis of stability analysis of the milling process under consideration. Stability boundary curve is tracked as the locus of points on the plane of the cutting parameters (Ω and w) at which maximum magnitude of the eigenvalues is one.

4. Results

Making use of Fargue approximation parameter $n=325$, principal period $T=\tau$ and dividing integer for piecewise constant approximation $k=10$, the stability chart generated for the system under study with parameters; $m=0.431kg\,,\omega_n=\frac{5700rad}{sec}$ and $\xi=0.02\,,C=3.5\times10^7\,\mathrm{Nm}^{\frac{-7}{4}}$ and $\mathrm{v}=0.0025\mathrm{m}/\sec$ is as shown in figure 6.

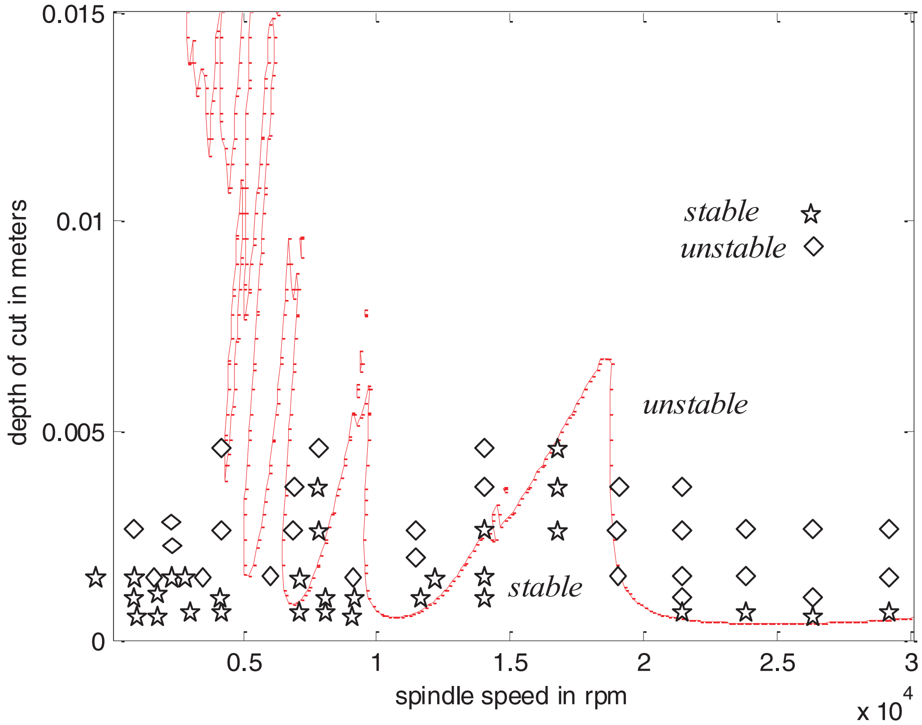


Figure6 : Plastic milling CNC machine stability chart using $n = 325$ *and* $k = 10$

First limitation of this study is that the resulting stability chart are time costly in that it takes not less than 38 hours of computation time of a modern laptop with processing speed of 2.10Gh. Another obvious limitation of this work is poor reliability of the stability chart at spindle speeds below 5000rpm. This stems from the method of stability analysis used. On this chart is a delineation the stable cutting domains from the unstable ones on the plane of cutting parameters; spindle speed Ω and depth of cut w. The region of the chart below the boundary curve is the stable region while the one above it is the unstable region as shown labelled in figure6. Sample stable operating conditions as validated by MATLAB dde23 solution of equation (16) are shown marked with star while the unstable (chatter) conditions are marked with diamond on the stability chart. The close agreement between MATLAB time histories of equation (16) with the chart is a testimony of its accuracy.

5. Conclusion

The operational spindle speed range of $(0 - 3000rpm)$ of the studied system falls within the poorly approximated low spindle speed domain of the chart. It will be noticed from the chart that chatter will be very unlikely for a depth of cut not more than 1mm. This result is in line with the specification on the actual machine. This could explain why laboratory operation became noisy at spindle speed of 1500rpm and depth of cut of 1.5mm. If the manufacturers and designers of the plastic end milling CNC machine under study were to provide operators and programmers of the machine with this chart, they (the manufacturers and designers) would be free to increase the power rating of the machine to achieve higher spindle speed range that allows for stable operation at deeper depths of cut. For example, under full immersion condition the machine operation at spindle speed $\Omega = 9000\text{rpm}$ and depth of cut $w = 3\text{mm}$

or spindle speed $\Omega = 17500\mathrm{rpm}$ and depth of cut $w = 4\mathrm{mm}$ is asymptotically stable. This chart is thus seen to have the potential to culminate in improved productivity of the machine from the point of design to usage.

References

Butcher, E., & Mann, B. P. *Stability Analysis and Control of Linear Periodic Delayed Systems using Chebyshev and Temporal Finite Element Methods.* Retrieved July 6, 2011, from http://mae.nmsu.edu/faculty/eab/bookchapter_final.pdf

Insperger, T. *Stability Analysis of Periodic Delay-Differential Equations Modelling Machine Tool Chatter* (*Doctoral dissertation, Budapest University of Technology and Economics, 2002*).

Insperger, T., Mann, B. P., Stepan, G., & Bayly, P. V. (2006). Stability of up-milling and down-milling, part 1: alternative analytical methods. International Journal of Machine Tools & Manufacture 43, 25–34.

Insperger, T., & Stepan, G. (2000). Stability of Milling Process. PERIODICA POLYTECHNICA SER. MECH. ENG. 44(1), 47–57.

Insperger, T., & Stepan, G. (2002). Semi-discretization method for delayed systems. International Journal For Numerical Methods In Engineering, 55, 503–518. doi: 10.1002/nme.505

National Institute of Standards and Technology. (1999) *High-Speed Machining Processes: Dynamics of Multiple Scales,* (Pub. No. 20899), 100 Bureau Drive, Gaithersburg USA: Davies, M.A., Burns, T. J., & Schmitz, T. L.

Ozoegwu, C. G. (2011). *Chatter of Plastic Milling CNC Machine.* Unpublished Master's thesis, Nnamdi Azikiwe University Awka, Nigeria.

Stepan, G. (1998). Delay-differential Equation Models for Machine Tool Chatter: In F. C. Moon (Ed.), *Nonlinear Dynamics of Material Processing and Manufacturing* (p. 165-192). New York: John Wiley & Sons.

Stépán, G., Szalai, R. & Insperger, T. (2003). Nonlinear Dynamics of High-Speed Milling Subjected to Regenerative Effect: In Gunther Radons (Ed.), *Nonlinear Dynamics of Production Systems* (pp. 1-2). New York: Wiley-VCH.